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COMMENT

The operator content of the ferromagnetic and antiferromagnetic spin-1 Zamolodchikov–Fateev quantum chain

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Abstract. The finite-size scaling spectra of the spin-1 Zamolodchikov–Fateev chain for an even and odd number of sites are presented. The operator content is given for free as well as toroidal boundary conditions. In the case of the antiferromagnetic chain we give separately the ferromagnetic and antiferromagnetic excitations.

The Zamolodchikov–Fateev spin-1 quantum chain is defined by the Hamiltonian:

$$H = \mathcal{N}(\lambda)\sum_{j=1}^{N}\{T_j - T_j^2 - 2(\cos\lambda - 1)(T_j^{\perp}T_j^z + T_j^z T_j^{\perp})\}$$
$$- 2\sin^2\lambda\,[T_j^z - (T_j^z)^2 + 2(S_j^z)^2] \qquad (0 \le \lambda < \pi/2) \tag{1}$$

where

$$T_j = T_j^{\perp} + T_j^z \qquad T_j^{\perp} = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \qquad T_j^z = S_j^z S_{j+1}^z \tag{2}$$

and S_j^x, S_j^y, S_j^z matrices give the spin-1 representation of SO(3). The normalisation factor $\mathcal{N}(\lambda)$ is different for the ferromagnetic and antiferromagnetic case:

$$\mathcal{N}(\lambda) = \begin{cases} \frac{-(\pi-\lambda)}{\pi\sin 2\lambda} & \text{ferromagnetic} \\ \frac{\lambda}{\pi\sin 2\lambda} & \text{antiferromagnetic} \end{cases} \qquad \lambda \neq 0. \tag{3}$$

Various properties of the spectra of the Hamiltonian (1) and their finite-scaling limit have been previously studied using both analytic and numerical methods (Zamolodchikov and Fateev 1980, di Francesco *et al* 1988, Alcaraz and Martins 1989, and references therein). It is the aim of this paper to give a conjecture for the whole operator content of the model with free and toroidal boundary conditions (BC). This conjecture is based on all the known results as well as on our own numerical studies of chains up to 14 sites. The Hamiltonian (1) can be generalised to arbitrary spins (Sogo *et al* 1983, Babujian and Tsvelick 1986, Kirilov and Reshetikhin 1987a, b) and the

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conformal structure of the spin-1 case will reveal what has to be expected for higher spin. The operator content for the spin-$\frac{1}{2}$ Heisenberg chain is already known (Alcaraz *et al* 1988).

The global symmetry of the infinite system is O(2), the Hamiltonian (1) being invariant under the transformations

$$(S_j^m)' = \sum_{n=-1}^{1} A^{mn} S_j^n \qquad S_j^{\pm 1} = S_j^x \pm i S_j^y \quad S_j^o = S_j^z \tag{4}$$

where the 3x3 matrices A^{mn} form the O(2) group

$$O(2) \simeq \{G(\theta)C^\alpha | \theta \in [0, 2\pi) \,,\ \alpha = 0, 1\} \tag{5}$$

where

$$G(\theta) = \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

We now specify the boundary conditions of the Hamiltonian (1):

(a) Free boundary conditions (H^F)

$$S_{N+1}^m = 0 \quad (m = 0, \pm 1) \tag{7}$$

(b) Toroidal boundary conditions (H^B)

$$S_{N+1}^m = \sum_{n=-1}^{1} B^{mn} S_1^n \tag{8}$$

where B is one of the matrices (6). Since two Hamiltonians H^{B_1} and H^{B_2} have the same spectrum if the group elements B_1 and B_2 belong to the same conjugacy class, we list the conjugacy classes of O(2):

$$\begin{array}{ll} \text{(I)} & \{\mathbf{1}\} \\ \text{(II)} & \{\mathbf{G}(\pi)\} \\ \text{(III)} & \{\mathbf{G}(\theta), G(-\theta)\} \bigvee \theta \neq 0, \pi \\ \text{(IV)} & \{\mathbf{G}(\theta)C | \theta \in [0, 2\pi)\}. \end{array} \tag{9}$$

The global symmetry of the Hamiltonian with free (H^F), periodic ($B = \mathbf{1}$) and antiperiodic ($B = G(\pi)$) BC is O(2). The symmetry for the elements of the conjugacy classes (III) is SO(2). In all these cases the 'charge' operator

$$\hat{Q} = \sum_{j=1}^{N} S_j^o \tag{10}$$

with eigenvalues Q commutes with the Hamiltonian. If the BC is given by one of the elements of the conjugacy class (IV) the symmetry is $Z_2 \otimes Z_2$. In what follows we will take as representative for this class the element with $\theta = 0$: $B = C$; in this case the

$Z_2 \otimes Z_2$ group is generated by C and $G(\pi)$. It is convenient to study together the BC belonging to the conjugacy classes (I), (II) and (III):

$$S^{\pm 1}_{N+1} = e^{2\pi i \ell} S^{\pm 1}_1 (0 \leq \ell < 1) \qquad S^0_{N+1} = S^0_1. \tag{11}$$

We now give the operator contents. We start with the ferromagnetic case (to our knowledge this case has not been up to now considered in the literature). In order to compute the finite-size limit spectra one does not have to distinguish between the cases of even and odd number of sites (this is different for the antiferromagnetic chain). The central charge of the Virasoro algebra is $c = 1$ and for the toroidal BC (11) one has the following partition function in the charge sector Q (Q integer):

$$\mathscr{E}^{\ell}_Q = \mathrm{Tr}(z^{L_0}\bar{z}^{\bar{L}_0}) = \sum_{n=-\infty}^{\infty} z^{[Q+4h(\ell+n)]^2/4h} \bar{z}^{[Q-4h(\ell+n)]^2/4h} \Pi_v(z)\Pi_v(\bar{z}) \tag{12}$$

where

$$h = \frac{\pi}{4\lambda} \qquad (\tfrac{1}{2} < h < \infty)$$
$$\Pi_v(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-1} \tag{13}$$

and $L_0, \bar{L}_0$ are generators of Virasoro algebras. This expression coincides with the operator content of the spin-$\frac{1}{2}$ chain (see Alcaraz *et al* 1988, equation (15)) with two differences: there the domain of h was different ($\frac{1}{4} \leq h < \infty$) and Q could be half-integer (for an odd number of sites). The operator content for free BC and for BC corresponding to the conjugacy class (IV) are the same as for the spin-$\frac{1}{2}$ chains with the same two differences.

We consider now the antiferromagnetic case (see equation (3)). Here we have to distinguish between the cases of even and odd values of the charge Q and the number of sites:

$$Q = 2k + \alpha \qquad N = 2n + \beta \qquad (\alpha, \beta = 0, 1 \in Z_2). \tag{14}$$

The central charge of the Virasoro algebra is $c = 3/2$ and the operator content for free BC reads:

$$\mathscr{E}_{2k+\alpha;\beta}(z) = \mathrm{Tr}(z^{L_0}) = R_{\alpha+\beta}(z) z^{(2k+\alpha)^2/4h} \Pi_v(z) \tag{15}$$

where

$$h = \left[2(1 - 2\lambda/\pi)\right]^{-1} \qquad R_0(z) = \chi_0(z) \qquad R_1(z) = \chi_{1/2}(z). \tag{16}$$

$\chi_0(z)$ and $\chi_{1/2}(z)$ are the character expressions of the unitary irreps of the Virasoro algebra with $c = 1/2$ (Ising model) and highest weights $\Delta = 0$ and $1/2$ respectively. As in the $c = 1$ case (Dijkgraaf *et al* 1988, Baake *et al* 1988) for rational value of h the spectra can be organised in terms of character functions of extended algebras. Let us now give a few examples. For $h = 1/2$ we have

$$\mathscr{E}^0(z) = \sum_{Q=-\infty}^{\infty} \mathscr{E}_{Q,0}(z) = \chi_0^{\mathrm{SU}(2)} \qquad \mathscr{E}^1(z) = \sum_{Q=-\infty}^{\infty} \mathscr{E}_{Q,1}(z) = \chi_{1/2}^{\mathrm{SU}(2)} \tag{17}$$

where $\chi_\Delta^{\mathrm{SU}(2)}$ are the characters of the SU(2) Kac–Moody algebra with highest weight Δ (Goddard *et al* 1986). For $h = 1/4$ we have

$$\mathscr{E}^0(z) = \chi_0^{N=2} \qquad \mathscr{E}^1(z) = \chi_{1/4}^{N=2} \tag{18}$$

where the $\chi_\Delta^{N=2}$ correspond to the $N = 2$ superconformal case (Ravanini and Yang 1987). If we take $h = 1/10$ we have

$$\mathscr{E}^0(z) = \chi_0^{\mathrm{ZF}} + \chi_1^{\mathrm{ZF}} \qquad \mathscr{E}^1(z) = \chi_{1/2}^{\mathrm{ZF}} + \chi_{5/2}^{\mathrm{ZF}} \tag{19}$$

where $\chi_\Delta^{\mathrm{ZF}}$ are characters of the Zamolodchikov–Fateev algebra (Zamolodchikov and Fateev 1985, Qiu 1988, Ravanini and Yang 1988).

We now consider toroidal BC. In this case we have to distinguish between ferromagnetic excitations which start at momentum zero and the antiferromagnetic ones which start at a Z_N momentum $p = N/2$ (for N even) an $p = (N \pm 1)/2$ (for N odd). We will denote by $\gamma = 0$ the ferromagnetic excitations and by $\gamma = 1$ the antiferromagnetic ones. For the boundary condition (11), in the charge sector Q, an even or odd number of sites (see equation (14)) and an excitation γ we have the following partition function:

$$\mathscr{E}^{\ell,\gamma}_{2k+\alpha,\beta}(z,\bar z) = R^{\alpha+\gamma}_{\alpha+\beta}(z,\bar z)\mathscr{F}^{\ell,\alpha+\beta+\gamma}_{2k+\alpha}(z,\bar z) = \mathrm{Tr}(z^{L_0}\bar z^{\bar L_0}) \qquad (\alpha,\beta,\gamma, \in Z_2) \tag{20}$$

where

$$\mathscr{F}^{\ell,\delta}_Q(z,\bar z) = \sum_{n=-\infty}^{\infty} z^{[Q+2h(\delta+2n+2\ell)]^2/16h}\bar z^{[Q-2h(\delta+2n+2\ell)]^2/16h}\Pi_v(z)\Pi_v(\bar z) \tag{21}$$

$\delta = 0, 1$ and

$$\begin{aligned} &R^0_0(z,\bar z) = \chi_0\bar\chi_0 \\ &R^0_1 = R^1_0 = \chi_{1/16}\bar\chi_{1/16} \\ &R^1_1(z,\bar z) = \chi_0\bar\chi_{1/2} + \chi_{1/2}\bar\chi_0 \end{aligned} \tag{22}$$

where again $\chi_0, \chi_{1/16}, \chi_{1/2}$ are characters of the $c = 1/2$ Virasoro algebras. As in the $c = 1$ case where the partition function (12) can be used to derive partition functions for sytems with $c < 1$ (Alacaraz *et al* 1989, Pasquier and Saleur 1989), the partition functions (20) can be used to derive partition functions for the $N = 1$ superconformal series (Capelli 1987). This will be the subject of a separate publication (Baranowski *et al* 1990).

We finally consider the BC corresponding the conjugacy class (IV) taking B = C (see equations (6) and (8)). It is convenient to separate the spectra considering the sectors given by the charge conjugation operation C with eigenvalues $(-1)^\epsilon$ ($\epsilon = 0, 1$) and $G(\pi)$ with eigenvalues $(-1)^\delta$ ($\delta = 0, 1$). As for the spin-$\frac{1}{2}$ chain (Alcaraz *et al* 1988 equation (16)) we expect to be able to write the partition functions in terms of the characters $\chi_\Delta^{N=2,\mathrm{TW}}$, corresponding to the $N = 2$ twisted superconformal algebra, which coincide with those of the twisted SU(2) Kac–Moody algebra (Rittenberg and Schwimmer 1987 and references therein) since the operator content is independent of h. For c = 3/2 we have $\Delta = 1/8r (r = 1, 2)$ and the character expressions are

$$\chi_{1/8r}^{N=2,\mathrm{TW}}(z) = z^{1/8r}\prod_{m=1}^{\infty}\frac{(1+z^{m/2})}{(1-z^{m/2})}\sum_{n=-\infty}^{\infty}(-1)^n z^{n^2+n(1-r/2)}. \tag{23}$$

It is useful to consider levels with a separation given by integers:

$$\chi_{1/16}^{N=2,\mathrm{TW}}(z) = \Psi_{1/16}(z) + \Psi_{9/16}(z) \qquad \chi_{1/8}^{N=2,\mathrm{TW}}(z) = \Psi_{1/8}(z) + \Psi_{5/8}(z). \tag{24}$$

For the partition functions $\mathscr{G}^{\gamma}_{\epsilon,\delta,\beta}$ (see equation (14)) we have:

$$\begin{aligned}
\mathscr{G}^0_{000} + \mathscr{G}^0_{010} &= \mathscr{G}^1_{101} + \mathscr{G}^1_{111} = \Psi_{1/16}\overline{\Psi}_{1/16} + \Psi_{9/16}\overline{\Psi}_{9/16} \\
\mathscr{G}^0_{100} + \mathscr{G}^0_{110} &= \mathscr{G}^1_{001} + \mathscr{G}^1_{011} = \Psi_{1/16}\overline{\Psi}_{9/16} + \Psi_{9/16}\overline{\Psi}_{1/16} \\
\mathscr{G}^0_{001} &= \mathscr{G}^0_{011} = \mathscr{G}^1_{000} = \mathscr{G}^1_{010} = \Psi_{1/8}\overline{\Psi}_{1/8} + \Psi_{5/8}\overline{\Psi}_{5/8} \\
\mathscr{G}^0_{101} &= \mathscr{G}^0_{111} = \mathscr{G}^1_{100} = \mathscr{G}^1_{110} = \Psi_{1/8}\overline{\Psi}_{5/8} + \Psi_{5/8}\overline{\Psi}_{1/8}.
\end{aligned} \tag{25}$$

This completes the list of the operator content.

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